Energy transport in a one-dimensional granular gas

Ítalo'Ivo Lima Dias Pinto,[1] Alexandre Rosas,[1] and Katja Lindenberg[2]
[1]*Departamento de Física, CCEN, Universidade Federal da Paraíba, Caixa Postal 5008, 58059-900 João Pessoa, PB, Brazil*
[2]*Department of Chemistry and Biochemistry and Institute for Nonlinear Science, University of California, San Diego, La Jolla, California 92093-0340, USA*

(Received 10 February 2009; revised manuscript received 22 May 2009; published 23 June 2009)

We study heat conduction in one-dimensional granular gases. In particular, we consider two mechanisms of viscous dissipation during intergrain collisions. In one, the dissipative force is proportional to the grain's velocity and dissipates not only energy but also momentum. In the other, the dissipative force is proportional to the relative velocity of the grains and therefore conserves momentum even while dissipating energy. This allows us to explore the role of momentum conservation in the heat conduction properties of this one-dimensional nonlinear system. We find normal thermal conduction whether or not momentum is conserved.

DOI: 10.1103/PhysRevE.79.061307 PACS number(s): 45.70.−n, 05.20.Dd, 05.70.Ln, 83.10.Rs

I. INTRODUCTION

Energy transport in low-dimensional systems can be pathological in the sense that Fourier's law for heat conduction might break down [1–4]. In three-dimensional solids, the energy (heat) transported is governed by Fourier's law, which says that the heat flux J is proportional to the gradient of the temperature,

$$J = -\kappa \nabla T, \tag{1}$$

where κ is the thermal conductivity. The thermal conductivity in Fourier's law is independent of system size and of time. This equation assumes that a local equilibrium is established at each time, so that one can define the local energy flux $J(x,t)$ and temperature $T(x,t)$, when the temperature gradient lies along the x direction [4]. In one-dimensional systems, one might expect a similar equation, with the gradient replaced by the derivative of $T(x,t)$ with respect to x, and J simply being the energy flux per unit time. However, it has been observed that in many one-dimensional models the thermal conductivity varies with the system size N as $\kappa \sim N^{\alpha}$, with α lying between 0.32 and 0.44 [4–6]. Self-consistent mode coupling analysis for one-dimensional nonlinear chains [7] predicts a universal value of $\alpha=2/5$, while renormalization-group analysis based on hydrodynamical models supports a universality class with $\alpha=1/3$. Recently, Mai *et al.* [5,6] revisited the problem and obtained the same $\alpha=1/3$ behavior for one-dimensional nonlinear chains as obtained for the fluidlike systems. They argue that the dynamic equations are those of a fluid at *all* length scales even if the static order of the chain extends to very large N. They explain the disagreement with most numerical results as a consequence of the numerical difficulties in reaching asymptotic behavior, which requires extremely large N, but their conclusions have in turn been disputed [6]. Whether there is a single universality class or there are two, there is a connection between heat conduction and diffusion in one-dimensional systems [8], that is, normal diffusion leads to normal heat conductivity ($\kappa \sim N^0$), whereas anomalous transport is associated with $\kappa \sim N^{\alpha}$, with $\alpha > 0 (<0)$ for super(sub) diffusion. Thus, superdiffusion is associated with a divergent thermal conductivity.

Another key concept in the understanding of the thermal conductivity in one-dimensional (1D) systems is momentum conservation [4]. In fact, momentum conservation usually implies the divergence of the thermal conductivity in 1D [1,9], and yet it is not necessary for the occurrence of anomalous diffusion [10]. An exception to the momentum conservation rule is a chain of particles interacting via the nearest-neighbor potential $V(q_i - q_{i-1}) = 1 - \cos(q_i - q_{i-1})$, the so-called rotator model [11]. This momentum-conserving model exhibits normal transport behavior to very high numerical accuracy and is said to occur because the rotator model "cannot support a nonvanishing pressure, and thus infinite-wavelength phonons cannot carry energy" [10]. In [4] this behavior is ascribed to the periodicity of the potential and the associated independent jumps from valley to valley.

In this contribution, we discuss heat conduction in one-dimensional granular gases. Granular gases are dissipative systems; the associated energy loss usually being modeled by introducing a coefficient of restitution as a parameter in the description of granular collisions. Instead, we follow a more dynamical approach and introduce energy dissipation explicitly via viscous terms in the equations of motion for the granules. There are a number of different sources and descriptions of viscous effects in the literature [12–18]. The different ways in which they appear in the dynamical descriptions suits our particular focus of interest, which is the role of momentum conservation in the heat transport process. In particular, one way to introduce viscosity conserves momentum [19–22], while the other does not [18,23,24]. Interestingly, we establish that thermal conduction exhibits normal behavior in both cases, that is, that there is no divergence as the system size increases.

In Sec. II we describe the two dissipative models, and in Sec. III we present simulation results to characterize the transport of heat through the granular gas in both cases. We conclude with a short summary in Sec. IV.

II. DISSIPATIVE MODELS

We consider N identical granules constrained to move on a line between two walls at different temperatures. The granules move freely except during collisions either with the

walls or with one another. The system is a one-dimensional "granular gas" because the distance between the walls is much greater than the space occupied by the granules. The walls act as heat baths; that is, whenever a granule collides with a wall at temperature T, its energy is absorbed by the wall and it acquires a new velocity away from the wall according to the probability distribution [25]

$$P(v) = \frac{|v|}{T}\exp(-v^2/2k_BT). \tag{2}$$

Interparticle collisions are governed by the power-law potential

$$V(\delta_{k,k+1}) = \frac{a}{n}|\delta|_{k,k+1}^n, \quad \delta \le 0,$$

$$V(\delta_{k,k+1}) = 0, \quad \delta > 0. \tag{3}$$

Here $\delta_{k,k+1} \equiv y_{k+1} - y_k$, a is a prefactor determined by Young's modulus and Poisson's ratio, and the principal radius of curvature R of the surfaces at the point of contact [26,27], and y_k is the displacement of granule k from its position at the beginning of the collision. The exponent n is 5/2 for spheres (Hertz potential), it is 2 for cylinders and, in general, depends on geometry. We stress that the one-sided (only repulsion) granular potential even with $n=2$ is entirely different from a two-sided (repulsion and attraction) harmonic potential. The one sidedness of the potential leads to analytic complexities even in the dissipationless case [24,28,29] and even greater complexities in the presence of dissipation [18–20,24].

In this paper, we explore the *low-density* limit, which leads to enormous analytical and computational simplifications. The low-density feature of these approximations is implemented via the assumption that the collisions are always *binary* [30], that is, that only two granules at a time are members of any collision event and that at any moment of time there is at most one collision.

Our approach starts with the equations of motion of the particles labeled by index k, and so we write y_k as a function of time τ. It is convenient to deal with scaled position and time variables x_k and t related to the unscaled variables y_k and τ as follows:

$$y_k = \left(\frac{mv_0^2}{a}\right)^{1/n} x_k, \quad \tau = \frac{1}{v_0}\left(\frac{mv_0^2}{a}\right)^{1/n} t. \tag{4}$$

Here m is the mass of the granules, and the velocity v_0 is an arbitrary choice in terms of which other velocities are expressed. We also introduce a scaled friction coefficient,

$$\gamma = \frac{\tilde{\gamma}}{mv_0}\left(\frac{mv_0^2}{a}\right)^{1/n}, \tag{5}$$

where $\tilde{\gamma}$ is the friction coefficient for the unscaled variables. In the low-density limit, we only need to consider the equations of motion for two colliding granules $k=1,2$. Furthermore, in this paper we only consider cylindrical grains, which leads to considerable simplification while still capturing the important general features of the system that we seek to highlight. We stress that the one-sided granular potential (i.e., one with only repulsive interactions) even with $n=2$ is entirely different from a two-sided harmonic potential.

Consider a viscous force that is proportional to the relative velocity of the colliding granules. Such a force has been considered not only theoretically [12,19,20] but it has also been observed experimentally [21,22]. While one might be tempted to think of this force as arising because one grain rubs against the other, the actual mechanism is more complicated and involves the medium that surrounds the granules [21,22]. The exact mechanism is still a matter of conjecture. In any case, the appropriate equations of motion in this case are

$$\ddot{x}_1 = [-\gamma(\dot{x}_1 - \dot{x}_2) - (x_1 - x_2)]\theta(x_1 - x_2),$$

$$\ddot{x}_2 = [\gamma(\dot{x}_1 - \dot{x}_2) + (x_1 - x_2)]\theta(x_1 - x_2), \tag{6}$$

where a dot denotes a derivative with respect to t. The Heaviside function is defined as $\theta(x)=1$ for $x>0$, $\theta(x)=0$ for $x<0$, and $\theta(0)=1/2$. It ensures that the two particles interact only when in contact, that is, only when the particles are loaded. The postcollisional velocities (called u below) can be written in terms of the velocities of the two granules at the beginning of the collision (called v) as

$$u_1 = \frac{1}{2}[1 - e^{-\gamma t_0}]v_1 + \frac{1}{2}[1 + e^{-\gamma t_0}]v_2,$$

$$u_2 = \frac{1}{2}[1 + e^{-\gamma t_0}]v_1 + \frac{1}{2}[1 - e^{-\gamma t_0}]v_2, \tag{7}$$

where $t_0 = \pi/(\sqrt{2-\gamma^2})$ is the duration of the collision. Since $u_1+u_2=v_1+v_2$, the momentum of the center of mass is conserved.

A model in which the center-of-mass velocity is not conserved is one governed by the equations of motion [18,24],

$$\ddot{x}_1 = [-\gamma\dot{x}_1 - (x_1 - x_2)]\theta(x_1 - x_1),$$

$$\ddot{x}_2 = [-\gamma\dot{x}_2 + (x_1 - x_2)]\theta(x_1 - x_1). \tag{8}$$

Here one might think of the viscosity arising from an interaction with the medium. However, such an interaction would produce a damping term not only during a collision but also while the granules are moving independently between collisions. In the low-density limit, the granules would hardly collide before stopping entirely unless the viscosity is extremely low, and one can then not talk of heat transport along the one-dimensional system. Thus, this model, while perhaps not realistic in any sense, is a "toy model" in which momentum is not conserved while still capable of supporting energy transport and therefore relevant to our question. Indeed, we can again calculate the postcollision velocities,

$$u_1 = \frac{1}{2}[e^{-\gamma t_0} - e^{-\gamma t_0/2}]v_1 + \frac{1}{2}[e^{-\gamma t_0} + e^{-\gamma t_0/2}]v_2,$$

$$u_2 = \frac{1}{2}[e^{-\gamma t_0} + e^{-\gamma t_0/2}]v_1 + \frac{1}{2}[e^{-\gamma t_0} - e^{-\gamma t_0/2}]v_2, \tag{9}$$

where $t_0 = 2\pi/(\sqrt{8-\gamma^2})$ is again the duration of the collision. It is easy to verify that in this case the momentum of the center of mass decays from v_1+v_2 before a collision to

$$u_1 + u_2 = [v_1 + v_2]e^{-\gamma t_0}, \tag{10}$$

after the collision, so that the momentum is not conserved.

Caution must be exercised in the choice of the damping coefficient γ. If it is too large, energy acquired from either wall is simply dissipated before it crosses the system, in which case the problem changes from one of energy transfer from one wall to the other through the granular gas to that of two walls pumping energy into the granular "sink." To estimate the limit on the damping coefficient, we can imagine a sequence of collisions whereby a granule starting from the hot wall with kinetic energy E_0 collides with the next granule, which in turn collides with the next one, and so on, until the last granule, whose energy is E_N, collides with the cold wall. The change in the kinetic energy due to each collision in the momentum-conserving model is

$$\Delta E \equiv \frac{1}{2}(u_1^2 + u_2^2) - \frac{1}{2}(v_1^2 + v_2^2) = -\frac{1}{4}(1 - e^{-2\gamma t_0})(v_1 + v_2)^2, \tag{11}$$

and in the momentum nonconserving model it is

$$\Delta E \equiv \frac{1}{2}(u_1^2 + u_2^2) - \frac{1}{2}(v_1^2 + v_2^2) = -\frac{1}{4}(1 - e^{-\gamma t_0})(v_1 + v_2)^2. \tag{12}$$

In both cases, the kinetic energy before and after the collision are related by an expression of the form $E_{after} = E_{before}(1 - b\gamma t_0)$ where b is a velocity-dependent dimensionless quantity of order unity, and where we have assumed that $\gamma t_0 \ll 1$. As an order of magnitude estimate, it is sufficient to write $E_N \sim E_0(1-\gamma t_0)^N$. The average energy of a hot granule is $k_B T_1/2$ and in order for energy to flow across the system the energy of the granule that arrives at the cold wall must be greater than $k_B T_2/2$. Consequently, we estimate that a requirement for energy transport is that

$$\gamma \lesssim 1 - \left(\frac{T_2}{T_1}\right)^{1/N}. \tag{13}$$

We now wish to explore the following specific questions. Is there a finite thermal conductivity for either or both of these models? If so, which model has a higher/lower thermal conductivity? How does the thermal conductivity depend on temperature? On viscosity? On system size? These are the questions we address in the next section.

III. NUMERICAL SIMULATIONS

The low-density limit allows us to use an event-driven algorithm in our simulations. As indicated earlier, when a granule collides with a wall at temperature T, its energy is absorbed by the wall and it acquires a new energy as determined by the velocity distribution given in Eq. (2). The initial temperature of the granular gas is arbitrarily set to $T=1$, and we allow the system to arrive at a steady state before beginning our "measurements." To ascertain that in the steady state we have achieved local equilibrium, we divide the system into bins and calculate the local temperature $\langle v^2 \rangle/2$ inside each bin by averaging v^2 over the granules inside the bin. We find these values to be stationary. Further evidence in support of a steady state can be seen, for example, in Fig. 1, where we show the energy absorbed and injected by each wall as a function of time for the dissipating momentum model. After a short transient, the rate of energy injection and absorption become constant, as they should in a steady state. We ran tests to ascertain that the initial temperature of the granular gas is not important for this equilibration; that is, we find that the rates of energy injection and absorption are independent of the initial velocity distribution of the granular gas. The momentum-conserving model equilibrates equally well, also independently of initial condition. Henceforth, we set the initial temperature of the granular gas to unity. In Fig. 2, we exhibit the steady-state temperature profiles of the momentum dissipating system (the momentum-conserving system shows similar behavior). The

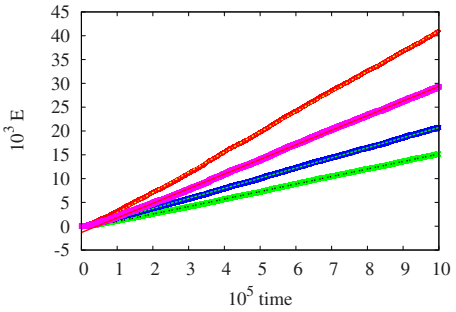

FIG. 1. (Color online) From top to bottom: energy injection by the hot wall (red), energy absorption by the cold wall (magenta), energy injection by the cold wall (blue), and energy absorption by the hot wall (green), for a momentum nonconserving system of $N=500$ particles between walls maintained at $T_l=6$ and $T_r=3$ and with $\gamma=0.0002$. The dashed lines are the linear regressions of these curves (after the transient). They yield the rate of energy injection/absorption. The initial temperature of the granular gas is $T=1$.

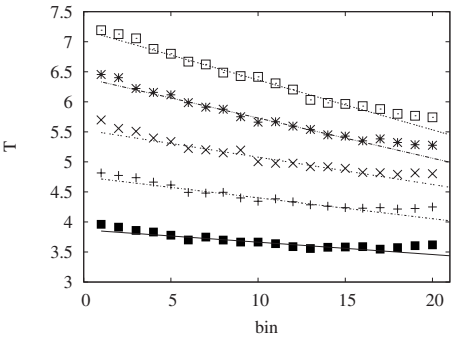

FIG. 2. Temperature profile in the granular gas at steady state. The temperature of the cold end is set to 3 and that of the hot wall is set to increasing values starting from 9 and going up in steps of 3. The viscosity coefficient $\gamma=0.0002$.

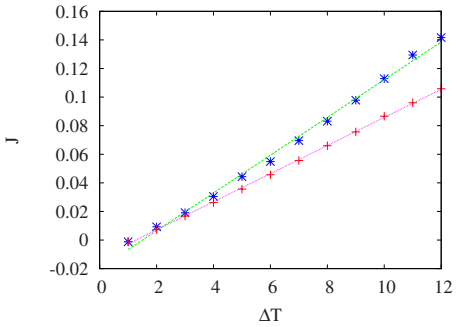

FIG. 3. (Color online) Rates of energy transmission as a function of the temperature gradient. In this simulation, the viscosity coefficient was set to $\gamma=0.0002$ and the system is composed of 500 granules. The (red) plus signs correspond to the momentum-conserving system, while the (blue) stars correspond to the momentum dissipating system. In this figure, the cold wall temperature was set to 3.

profiles are close to linear, especially in the interior of the system. While ideally one would want the temperature gradient in the steady state to be strictly linear as assumed in the linear-response theory that leads to the Fourier law, the subtleties encountered in 1D systems are well known and observed pretty much no matter how the thermalization is implemented [6,25] and the resulting gradients are, as in our systems, only strictly linear away from the boundaries. In any case, we can address the question of whether or not there are system size divergences in the transport of heat.

Next, we discuss the dependence of the energy flux on the temperature gradient. The energy flux $J=dE/dt$ was calculated as the slope of the transmitted energy as a function of time, that is, [(energy injection by the hot wall-energy absorption by the hot wall)-(energy injection by the cold wall-energy absorption by the cold wall)-(energy dissipated during flow along the chain)] per unit time. As seen in Fig. 3, both models lead to a behavior fairly well described by Fourier's law. As an aside, we note that the heat current is seen in this figure to go to zero at a finite temperature. As noted in the discussion surrounding Eq. (13), γ must not be larger than the upper bound indicated there in order for the energy injected from the hot wall to reach the cold wall. In Fig. 3,

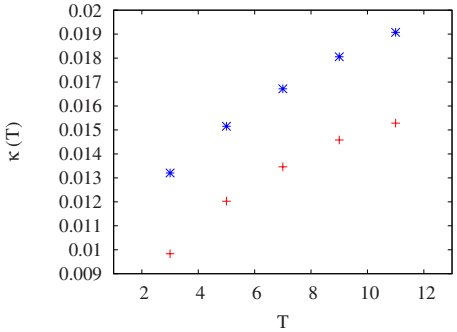

FIG. 4. (Color online) Thermal conductivity as a function of the temperature. In this simulation, the viscosity coefficient was set to $\gamma=0.0002$ and the system is composed of 500 granules. The (red) plus signs correspond to the momentum-conserving system; the (blue) stars to the momentum dissipating system.

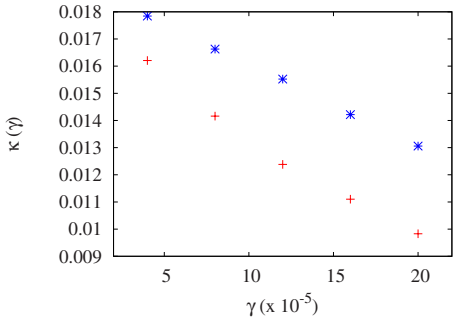

FIG. 5. (Color online) Thermal conductivity as a function of the viscosity coefficient. In this simulation, the temperature of the cold wall is set to 3 and that of the hot wall to 6. The system is composed of 500 granules. The (red) plus signs correspond to the momentum-conserving system; the (blue) stars to the momentum dissipating system.

when $\Delta T\rightarrow 1$ this argument sets the upper limit for γ at 0.0005. This value is sufficiently close to the value used in the figure for the flux to become very small. For smaller ΔT, all the energy injected into the system is dissipated before reaching the other wall and there is no heat conduction.

Repeating the plot of Fig. 3 for different temperatures of the cold wall, we obtain the dependence of the thermal conductivity on the temperature. As observed in Fig. 4, κ is an increasing function of T for both models, but it is larger for the momentum dissipating system [31]. On the other hand, for a given temperature, the thermal conductivity decreases (almost linearly) with the viscosity (see Fig. 5). Once again, the dissipating system presents larger values of κ.

Finally, we have analyzed the dependence of the rate of energy transmission on system size, for a fixed temperature gradient and fixed viscosity. This is the crucial test of normal vs anomalous behavior. In Fig. 6, we can clearly see that as the size of the system increases, the rate of energy transmission does not increase with system size. In fact, it *decreases*. Therefore, in our model the thermal conductivity does not diverge with increasing system size regardless of momentum conservation or dissipation. One may be tempted to think of this behavior entirely as a consequence of the energy dissi-

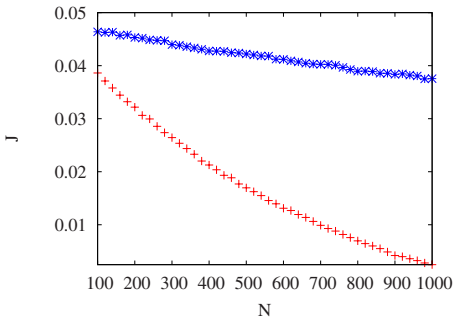

FIG. 6. (Color online) Rates of energy transmission as a function of the system size. In this simulation, the viscosity coefficient was set to $\gamma=0.0001$ and the temperatures of the walls were set to 3 and 6. The (red) plus signs correspond to the momentum-conserving system; the (blue) stars to the momentum dissipating system.

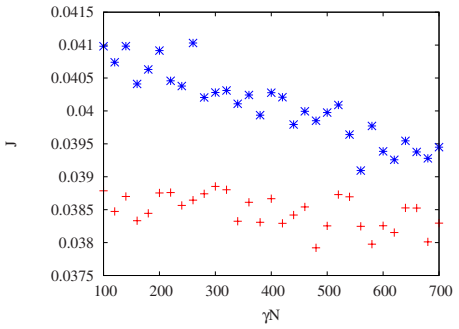

FIG. 7. (Color online) Rates of energy transmission as a function of the scaled system size. The temperatures of the walls were set to 3 and 6. The (red) plus signs correspond to the momentum-conserving system; the (blue) stars to the momentum dissipating system.

pation because for bigger systems the number of collisions necessary for the energy to be transmitted from one end of the system to the other increases. Hence, more energy is dissipated and less energy arrives at the cold wall. However, the introduction of dissipation changes the dynamics more profoundly. This can be seen in Fig. 7, where we plot the rates of energy transmission against γN, thus, taking into account the "simple" effect of energy dissipation [see the discussion preceding Eq. (13)]. The rate of energy transfer in the momentum-conserving system is essentially independent of system size in this scaled representation but that of the momentum dissipating system actually decreases even when scaled in this way. In any case, we note the very small scale of variation in the ordinate in Fig. 7.

IV. CONCLUSIONS

Heat transport in one-dimensional discrete systems continues to be a problem of theoretical interest, uncertainty, and even controversy [1–6]. Very recently, experimental results in this arena have also started to appear. In [32] the focus is the understanding of transport of heat along colliding granular beads in a liquid medium where the questions of interest involve the nature of the contact regions ("liquid bridges") between granules. In [33] the issue is the breakdown of Fourier's law in nanotube thermal conductors in that the thermal conductivity diverges with length of the nanotube. The point is that even after many years of study, the conditions that lead to the validity or violation of Fourier's law are not yet clear. While momentum conservation has often been featured as a condition closely associated with the system size divergence of the 1D thermal conductivity, we have examined two dissipative granular gases, one of which involves momentum dissipation along with energy dissipation, whereas in the other momentum is conserved. In both of these, the heat flux remains finite and in fact decreases as the number of granules in the system increases, as it must in a system where each collision leads to energy dissipation. This in turn indicates that the thermal conductivity does not grow with system size. In the momentum dissipating model, the heat flux decreases with increasing system size more rapidly than can be accounted for by the simply heat loss mechanism of the collisions, indicating a more profound change in the dynamics. This behavior points to the caution that must be exercised when associating momentum conservation with anomalous behavior in one dimension.

ACKNOWLEDGMENTS

A.R. and I.L.D.P. acknowledge support from Pronex-CNPq-FAPESQ and CNPq. Acknowledgment is made to the Donors of the American Chemical Society Petroleum Research Fund for partial support of this research (K.L.).

[1] O. Narayan and S. Ramaswamy, Phys. Rev. Lett. **89**, 200601 (2002).
[2] S. Lepri, R. Livi, and A. Politi, Phys. Rev. E **68**, 067102 (2003).
[3] A. Dhar and K. Saito, Phys. Rev. E **78**, 061136 (2008).
[4] S. Lepri, R. Livi, and A. Politi, Phys. Rep. **377**, 1 (2003).
[5] T. Mai and O. Narayan, Phys. Rev. E **73**, 061202 (2006).
[6] T. Mai, A. Dhar, and O. Narayan, Phys. Rev. Lett. **98**, 184301 (2007); L. Delfini, S. Lepri, R. Livi, and A. Politi, *ibid.* **100**, 199401 (2008); A. Dhar and O. Narayan, *ibid.* **100**, 199402 (2008).
[7] S. Lepri, R. Livi, and A. Politi, Europhys. Lett. **43**, 271 (1998).
[8] B. Li and J. Wang, Phys. Rev. Lett. **91**, 044301 (2003); B. Li, J. Wang, L. Wang, and G. Zhang, Chaos **15**, 015121 (2005).
[9] T. Prosen and D. K. Campbell, Phys. Rev. Lett. **84**, 2857 (2000).
[10] T. Prosen and D. K. Campbell, Chaos **15**, 015117 (2005).
[11] C. S. Giardiná, R. Livi, A. Politi, and M. Vassalli, Phys. Rev. Lett. **84**, 2144 (2000).
[12] G. E. Duvall, R. Manvi, and S. C. Lowell, J. Appl. Phys. **40**, 3771 (1969).
[13] C. Brunhuber, F. G. Mertens, and Y. Gaididei, Phys. Rev. E **73**, 016614 (2006).
[14] E. Arevalo, Y. Gaididei, and F. G. Mertens, Eur. Phys. J. B **27**, 63 (2002).
[15] N. V. Brilliantov, F. Spahn, J. M. Hertzsch, and T. Poschel, Phys. Rev. E **53**, 5382 (1996); R. Ramirez, T. Pöschel, N. V. Brilliantov, and T. Schwager, *ibid.* **60**, 4465 (1999); T. Pöschel and N. V. Brilliantov, in *Lecture Notes in Physics: Granular Gases*, edited by S. Luding (Springer, New York, 2000), Vol. 564, pp. 203–212.
[16] M. Manciu, S. Sen, and A. J. Hurd, Physica D **157**, 226 (2001); S. Sen *et al.*, in *AIP Conference Proceedings: Modern Challenges in Statistical Mechanics*, edited by V. M. Kenkre and K. Lindenberg (AIP, New York, 2003), Vol. 658, pp. 357–379.
[17] S. McNamara and E. Falcon, Phys. Rev. E **71**, 031302 (2005).
[18] A. Rosas, J. Buceta, and K. Lindenberg, Phys. Rev. E **68**, 021303 (2003).

[19] A. Rosas, A. H. Romero, V. F. Nesterenko, and K. Lindenberg, Phys. Rev. Lett. **98**, 164301 (2007).
[20] A. Rosas, A. H. Romero, V. F. Nesterenko, and K. Lindenberg, Phys. Rev. E **78**, 051303 (2008).
[21] E. B. Herbold, V. F. Nesterenko, and C. Daraio, in *Shock Compression of Condensed Matter, 2005*, AIP Conference Proceeding Vol. 845 (AIP, New York, 2006), pp. 1523–1526; or e-print arXiv:cond-mat/0512367.
[22] E. B. Herbold and V. F. Nesterenko, Phys. Rev. E **75**, 021304 (2007).
[23] A. Rosas, D. ben-Avraham, and K. Lindenberg, Phys. Rev. E **71**, 032301 (2005).
[24] A. Rosas and K. Lindenberg, Phys. Rev. E **68**, 041304 (2003).
[25] R. Tehver, F. Toigo, J. Koplik, and J. R. Banavar, Phys. Rev. E **57**, R17 (1998).
[26] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Addison-Wesley, Massachusetts, 1959), p. 30.
[27] H. Hertz, J. Reine Angew. Math. **92**, 156 (1881).
[28] V. F. Nesterenko, J. Appl. Mech. Tech. Phys. **24**, 733 (1983).
[29] E. J. Hinch and S. Saint-Jean, Proc. R. Soc. London, Ser. A **455**, 3201 (1999).
[30] A. Rosas and K. Lindenberg, Phys. Rev. E **69**, 037601 (2004).
[31] N. Li and B. Li, EPL **78**, 34001 (2007).
[32] J.-C. Géminard, D. Bouraya, and H. Gayvallet, Eur. Phys. J. B **48**, 509 (2005).
[33] C. W. Chang, D. Okawa, H. Garcia, A. Majumdar, and A. Zettl, Phys. Rev. Lett. **101**, 075903 (2008).